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ON TWO-DIMENSIONAL LAMINAR FLOWS OF A PARTICULATE SUSPENSION IN THE PRESENCE OF GRAVITY FIELD

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Abstract—Three simple two-dimensional streaming motions of a mixture of solid particles with a continuous carrier fluid, or gas, in the presence of the gravity field are considered. These include flow of a mixture over an infinite stationary rigid plane perpendicular to the direction of the gravity field, flow near an oscillating rigid plane and flow in a mixture induced by a suddenly accelerated plane. The nature of the boundary conditions at the interface between a layer of sediment settling on the rigid boundary and the mixture above it suggests an introduction of the independent variables that enable simple analytical expressions for the solutions of the first two flows and a numerical solution by means of a Laplace transform in the last case.

1. INTRODUCTION

We consider three simple two-dimensional flows of a mixture of continuous fluid or gas with a cloud of spherical solid particles of approximately equal size, occurring in the presence of a gravity field. All three flows are induced by the boundary conditions introduced by the presence of the rigid boundary—an infinite plane with a unit normal antiparallel to the direction of the gravity field. Considered flow situations are counterparts of a single-phase flow over a stationary rigid plane, flow near an oscillating plane and flow induced in a fluid at rest by a suddenly accelerated plane.

An attempt has been made to consider these simple flows on the basis of a continuum model of a suspension of solid particles in a carrier fluid; see e.g. Di Giovanni & Lee (1974), Ishii (1975) and Drew (1979, 1983). In these theories, the constituents are treated as superimposed continua and described by means of field variables and balance equations, obtained through an averaging procedure over regions containing sufficiently large numbers of solid particles. We thus write separate mass and momentum balance equations for each constituent of a mixture. The momentum equations for each phase are coupled through a fluid–solid interaction force, which is taken to be the classical Stokes drag on a single spherical particle and modified by a factor, accounting for a finite volume fraction of particles, obtained by Tam (1969).

The presence of the gravity field introduces separational motion of the phases in the direction perpendicular to the rigid plane, resulting in an origin of a growing layer of dense sediment on the plane. Considering the mass balance across the interface between this layer of sediment and the mixture above it, we obtain an expression describing the growth velocity. Further, we assume that the layer of sediment has the same horizontal motion as the rigid boundary and formulate thus boundary conditions for the mixture at the interface between the sediment and the mixture. The character of the boundary conditions suggests an introduction of a coordinate system with a horizontal axis coinciding with the interface, and thus propagating in the vertical direction with the velocity of the interface. Introducing accordingly new independent variables in an appropriate way, we simplify the momentum equations, which then allow simple analytical expressions for the solutions in the first two cases. Solution of the third flow situation is obtained by means of a numerical inversion of the solutions to the Laplace transformed momentum equations.

2. CONSERVATION EQUATIONS

We are considering here certain two-dimensional flows of a mixture of solid spherical particles homogeneously distributed in an incompressible carrier fluid, under the influence of a gravity field. The common condition for these flows is that the velocity component of the mixture at infinity, or the velocity of a moving boundary, is perpendicular to the direction of the gravity field, which introduces separational motion of the phases.

The mixture is treated by a continuum approach to both constituents. We thus write separate equations of mass conservation and momentum balance for each phase. These average balance equations are formulated in terms of the following averaged variables: volume fraction of the dispersed phase α, phase velocities v_k^d and v_k^c, pressure p^c and stress tensor of the continuous phase τ_{rk}^c. The stress tensor of the dispersed phase is taken to be zero (see Drew 1983).

$$\alpha_{,t} + (\alpha v_k^d)_{,k} = 0, \qquad [1]$$

$$(1-\alpha)_{,t} + [(1-\alpha)v_k^c]_{,k} = 0, \qquad [2]$$

$$\alpha\rho^d(v_{k,t}^d + v_{k,r}^d v_r^d) = \alpha f_k^d - \alpha p_{,k}^c + M_k^d, \qquad [3]$$

$$(1-\alpha)\rho^c(v_{k,t}^c + v_{k,r}^c v_r^c) = (1-\alpha) f_k^c - (1-\alpha)\, p_{,k}^c + [(1-\alpha)\tau_{rk}^c]_{,r} + M_k^c, \qquad [4]$$

where f_k^d and f_k^c are the body forces per unit volume and M_k^d and M_k^c are the fluid–solid interaction forces per unit volume and are of the form

$$M_k^d = f(\alpha)F_k, \qquad [5]$$

$$M_k^c = -f(\alpha)F_k, \qquad [6]$$

where F_k is the generalized drag force acting on a single particle and may include, besides the classical Stokes' drag, the shear-lift force, acting on a particle of the dispersed phase in a uniform shear field of the continuous fluid (see Saffman 1965), the virtual mass term (Zuber 1964) and the spin-lift force induced by the inner rotation of a particle (Rubinow & Keller 1961). Each of these forces, acting on a single particle, is modified by its own correction factor $f(\alpha)$, accounting for the finite volume fraction of the dispersed phase. Under the present assumption of small-particle Reynolds number, however, the leading term in the generalized drag force is the classical Stokes' drag. All other drag forces will therefore be neglected in the present investigation. We thus may write

$$F_k = \kappa \frac{\mu_c}{a^2} f(\alpha)\,(v_k^c - v_k^d), \qquad [7]$$

where $\kappa = 4.5$ and $f(\alpha)$ is the correction factor accounting for the finite volume fraction of the dispersed phase, obtained by Tam (1969) in the case of spherical particles, and is of the form (per unit volume of the flow field)

$$f(\alpha) = \frac{4 + 3(8\alpha - 3\alpha^2)^{1/2} + 3\alpha}{(2-3\alpha)^2}\alpha. \qquad [8]$$

In the case of a two-dimensional flow of a mixture over a flat plate of infinite length and with the plane normal parallel to the direction of the gravity field, the flow variables are assumed to be functions of the vertical coordinate y and time t.

We next introduce a set of dimensionless variables, using the notations of Greenspan (1983),

$$y_* = \frac{y}{a}, \qquad [9]$$

$$\epsilon = \frac{\rho_d - \rho_c}{\rho_c}, \qquad [10]$$

$$t_* = \frac{ga|\epsilon|}{\kappa\nu_c} t, \qquad [11]$$

and seek the solution in the form

$$\bar{v}_d = \frac{ga^2|\epsilon|}{\kappa\nu_c}[U_d(t_*, y_*)\bar{e}_x + V_d(t_*, y_*)\bar{e}_y], \qquad [12]$$

$$\bar{v}_c = \frac{ga^2|\epsilon|}{\kappa\nu_c}[U_c(t_*, y_*)\bar{e}_x + V_c(t_*, y_*)\bar{e}_y], \qquad [13]$$

$$p_c = -\rho_c g y [1 - |\epsilon| P(t_*)]. \qquad [14]$$

Dropping the asterisk notation, we write the dimensionless form of the conservation equations of mass for each phase,

d: $$\alpha_t + (\alpha V_d)_y = 0, \qquad [15]$$

c: $$-\alpha_t + [(1-\alpha)V_c]_y = 0, \qquad [16]$$

and momentum for each phase,

$d, \bar{e}_x$: $$(1+\epsilon)\beta[U_{dt} + V_d U_{dy}] = \frac{f(\alpha)}{\alpha}(U_c - U_d), \qquad [17]$$

$c, \bar{e}_x$: $$\beta[U_{ct} + V_c U_{cy}] = \frac{1}{\kappa}U_{cyy} - \frac{f(\alpha)}{1-\alpha}(U_c - U_d), \qquad [18]$$

$d, \bar{e}_y$: $$(1+\epsilon)\beta[V_{dt} + V_d V_{dy}] = -\frac{\epsilon}{|\epsilon|} + P + \frac{f(\alpha)}{\alpha}(V_c - V_d), \qquad [19]$$

$c, \bar{e}_y$: $$\beta[V_{ct} + V_c V_{cy}] = P + \frac{1}{\kappa}V_{cyy} + \frac{f(\alpha)}{1-\alpha}(V_c - V_d), \qquad [20]$$

where

$$\beta = \frac{ga^2|\epsilon|}{\kappa\nu_c}\frac{a}{\kappa\nu_c} \qquad [21]$$

is the Reynolds number based on the particle size and the viscosity of the continuous phase.

3. STEADY FLOW OF A MIXTURE OVER AN INFINITE PLANE

We assume that the mixture of solid particles with a fluid flows over a solid boundary—an infinite plane $y = 0$ with a unit normal antiparallel to the direction of the gravity field $\bar{g}$. We assume further that there is not net flow in the direction parallel to the direction of gravity field, which introduces separational motion of the phases. Such a process is called batch sedimentation (see e.g. Wallis 1969). According to Kynch (1952), the vertical sedimentation of solid particles may proceed in three varying ways, depending on the shape of the curve of the total particle flow rate versus the volumetric fraction of the particles. One of the possible ways is when a direct shock from the initial value of the concentration α to the final fully settled value α_M is formed at the interface of a mixture and maximally concentrated dispersed phase, settled at the horizontal solid plane. In other cases, a region with a nonuniform concentration of the dispersed phase may be formed between the mixture with the initial concentration of particles at the top and maximally concentrated dispersed phase at the bottom. We are here going to investigate the first case, since it is easier to analyze but also quite common. We thus assume that the initial concentration of the dispersed phase is uniform and remains so throughout the process, in the region occupied by the mixture; in other words

$$\alpha = \text{const.} \qquad [22]$$

Equations of continuity for each phase give therefore

$$V_{dy} = V_{cy} = 0, \qquad [23]$$

which simplifies the y-components of the momentum equations:

$$d, \bar{e}_y: \qquad (1+\epsilon)\beta V_{dt} = -\frac{\epsilon}{|\epsilon|} + P + \frac{f(\alpha)}{\alpha}(V_c - V_d), \qquad [24]$$

$$c, \bar{e}_y: \qquad \beta V_{ct} = P - \frac{f(\alpha)}{1-\alpha}(V_c - V_d). \qquad [25]$$

Since the overall volumetric flux in the y-direction is zero (batch sedimentation), we have

$$\alpha V_d + (1-\alpha)V_c = 0. \qquad [26]$$

Equations [24]–[26] with initial conditions

$$V_d(0) = V_c(0) = 0 \qquad [27]$$

are readily integrated to give

$$V_d = -\frac{\epsilon}{|\epsilon|}\frac{\alpha(1-\alpha)^2}{f(\alpha)}\left[1 - \exp\left(-\frac{f(\alpha)}{\alpha(1-\alpha)^2\beta[1+\epsilon+\alpha/(1-\alpha)]}t\right)\right], \qquad [28]$$

$$V_c = \frac{\epsilon}{|\epsilon|}\frac{\alpha^2(1-\alpha)}{f(\alpha)}\left[1 - \exp\left(-\frac{f(\alpha)}{\alpha(1-\alpha)^2\beta[1+\epsilon+\alpha/(1-\alpha)]}t\right)\right], \qquad [29]$$

Plots of the vertical velocity components of both phases for various sets of flow variables are shown in figure 1, which illustrates the adjustment of the vertical motion of the phases, imposed by the gravity field, to its stationary state. This adjustment occurs over a transient period, with a duration depending on the density ratio between the phases, the concentration of the dispersed phase and the value of the particle Reynolds number. We will subsequently assume, however, that the vertical motion of the phases has reached its steady state, with the verticle components of the velocities given by

$$V_d = -\frac{\epsilon}{|\epsilon|}\frac{\alpha(1-\alpha)^2}{f(\alpha)}, \qquad [30]$$

$$V_c = \frac{\epsilon}{|\epsilon|}\frac{\alpha^2(1-\alpha)}{f(\alpha)}, \qquad [31]$$

Consider now the moving interface between the mixture at the top and the maximally concentrated dispersed phase at the bottom—a horizontal plane. Concentration of the dispersed phase in the mixture is α and in the region occupied by the sediment α_M (α_M is ~0.6 for spheres). Apply the conservation of mass for each component across the moving interface. If the velocity of the plane of discontinuity is V, then

$$d: \quad \alpha(V_d - V)\,|_-^+ = 0, \qquad [32]$$

$$c: \quad (1 - \alpha)(V_c - V)\,|_-^+ = 0. \qquad [33]$$

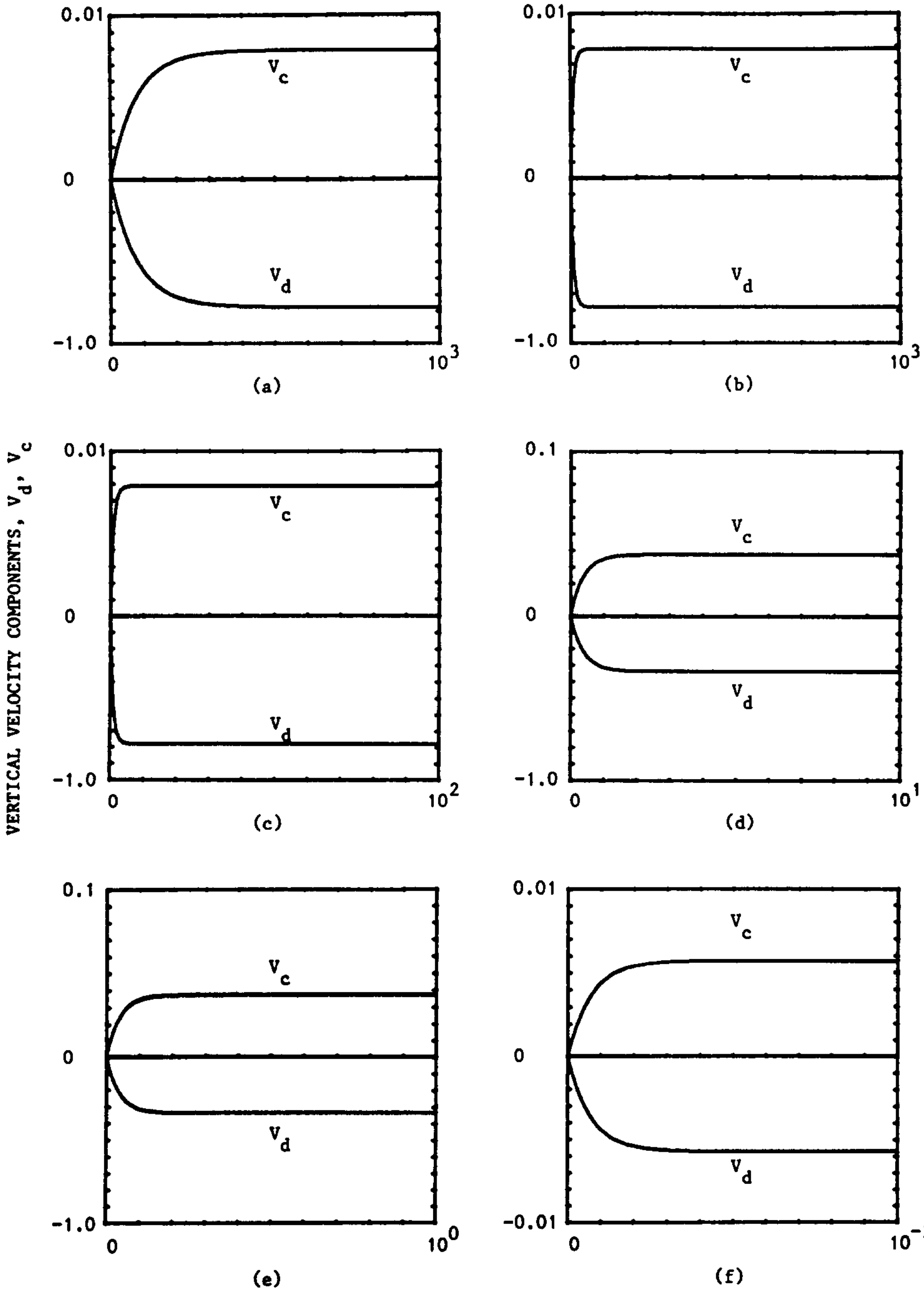

Figure 1. Vertical velocities of the phases versus time. (a) $\alpha = 0.01, \beta = 0.1, \epsilon = 1000$; (b) $\alpha = 0.01, \beta = 0.1, \epsilon = 100$; (c) $\alpha = 0.01, \beta = 0.1, \epsilon = 10$; (d) $\alpha = 0.1, \beta = 0.1, \epsilon = 10$; (e) $\alpha = 0.1, \beta = 0.01, \epsilon = 10$; (f) $\alpha = 0.5, \beta = 0.01, \epsilon = 10$.

At the interface between the mixture "+" and the sediment "−," we have

$$\alpha^+ = \alpha, \alpha^- = \alpha_M, V_d^+ = -\frac{\epsilon}{|\epsilon|}\frac{\alpha(1-\alpha)^2}{f(\alpha)}, V_d^- = 0, V_c^+ = \frac{\epsilon}{|\epsilon|}\frac{\alpha^2(1-\alpha)}{f(\alpha)}, V_c^- = 0, \quad [34]$$

which gives

$$V = \frac{1-\alpha}{\alpha_M - \alpha} V_c^+ = -\frac{\alpha}{\alpha_M - \alpha} V_d^+ = \frac{\epsilon}{|\epsilon|}\frac{\alpha^2(1-\alpha)^2}{(\alpha_M - \alpha) f(\alpha)} \quad [35]$$

Consider now the x-components of the momentum equations

$$d, \bar{e}_x: \quad (1+\epsilon)\beta[U_{dt} + V_d U_{dy}] = \frac{f(\alpha)}{\alpha}(U_c - U_d), \quad [36]$$

$$c, \bar{e}_x: \quad \beta[U_{ct} + V_c U_{cy}] = \frac{1}{\kappa} U_{cyy} - \frac{f(\alpha)}{1-\alpha}(U_c - U_d), \quad [37]$$

In this section, we are imposing the following boundary conditions on the horizontal components of the velocities. We will assume the "no-slip" condition for the horizontal component of the velocity of the continuous phase in the mixture at the moving interface

$$y = Vt, \quad [38]$$

between the mixture and the sediment at the bottom, which is assumed to be at rest with respect to the horizontal motion of the solid boundary $y = 0$. Introducing a grouping

$$\eta = y - Vt, \quad [39]$$

this boundary condition may be formulated as

$$U_c = 0, \qquad \eta = 0. \quad [40]$$

The horizontal velocity component of the continuous phase at infinity $y = +\infty$ is equal to U:

$$U_c = U, \qquad \eta = +\infty, \quad [41]$$

We are thus seeking the steady-state distribution of U_d and U_c, satisfying momentum equations and boundary conditions. In this case, the independent variables t and y will appear in the solution in the combination introduced in [39] only, and the governing differential equations may thus be reduced from one of partial to one of the ordinary type, resulting in the following system:

$$-(1+\epsilon)\beta(V - V_d)U_d' = \frac{f(\alpha)}{\alpha}(U_c - U_d), \quad [42]$$

$$-\beta(V - V_c)U_c' = \frac{1}{\kappa} U_c'' - \frac{f(\alpha)}{1-\alpha}(U_c - U_d). \quad [43]$$

Equations [42] and [43] form a system of ordinary differential equations with constant coefficients. The general solution of such a system is obtained by finding the roots of the characteristic equation, being in this case an algebraic equation of the third degree. All three roots of the equation are real:

$$\lambda_1 = 0, \lambda_2 > 0, \lambda_3 < 0. \quad [44]$$

Choosing $\lambda_1 = 0$ and $\lambda_3 = \lambda < 0$ (since $\eta \geq 0$ and determining the two arbitrary constants by means of [40] and [41], we obtain

$$U_c = U(1 - e^{\lambda\eta}), \qquad [45]$$

and further using [43]

$$U_d = U\left(1 - \left[1 - \beta(V - V_c)\frac{1-\alpha}{f(\alpha)}\lambda - \frac{1-\alpha}{\kappa f(\alpha)}\lambda^2\right]e^{\lambda\eta}\right). \qquad [46]$$

Plots of U_d and U_c for various values of flow parameters are shown in figure 2.

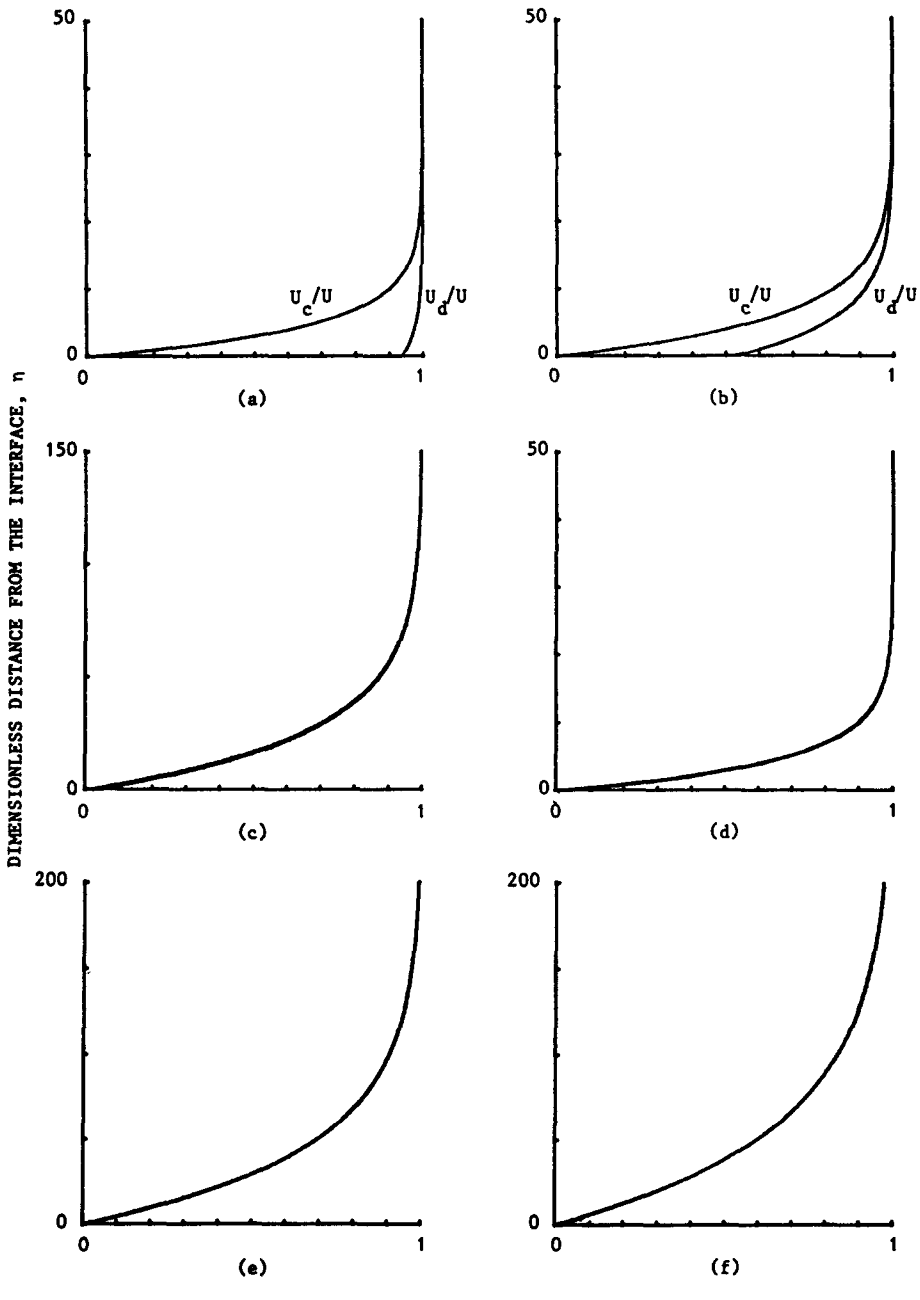

Figure 2. Steady flow over an infinite plane Horizontal velocity profiles of the phases. (a) $\alpha = 0.01$, $\beta = 0.1$, $\epsilon = 1000$; (b) $\alpha = 0.01$, $\beta = 0.1$, $\epsilon = 100$; (c) $\alpha = 0.01$, $\beta = 0.1$, $\epsilon = 10$; (d) $\alpha = 0.1$, $\beta = 0.1$, $\epsilon = 10$; (e) $\alpha = 0.1$, $\beta = 0.01$, $\epsilon = 10$; (f) $\alpha = 0.5$, $\beta = 0.01$, $\epsilon = 10$.

4. FLOW IN A MIXTURE DUE TO AN OSCILLATING INFINITE PLANE

In this section, we assume that the infinite flat wall $y = 0$ executes linear harmonic oscillations parallel to its own direction. We thus suppose that the motion of the plane is given by

$$U_{\text{plane}} = U \cos \omega t. \quad [47]$$

By analogy with the previous section, we assume that the dense sediment, settled down on the plane under the influence of the gravity field, having a direction perpendicular to the plane, is at rest with respect to the horizontal motion of the plane, and therefore acquires the horizontal velocity component defined in [47]. Consider the moving interface between the sediment and the mixture, propagating upward with the velocity V, obtained in the previous section. Here, besides the vertical motion due to the settling process, the interface acquires the horizontal motion of the solid plane [47]. As before, we will assume the "no-slip" conditon for the horizontal component of the velocity of the continuous phase at the moving interface,

$$U_c = U \cos \omega t, \qquad \eta = 0, \quad [48]$$

whereas in the previous section,

$$\eta = y - Vt. \quad [49]$$

Upon introduction of the new independent variables t and η, the x-components of the momentum equations [36] and [37] take the form

$$d\colon \quad (1 + \epsilon)\beta[U_{dt} - (V - V_d)U_{d\eta}] = \frac{f(\alpha)}{\alpha}(U_c - U_d), \quad [50]$$

$$c\colon \quad \beta[U_{ct} - (V - V_c)U_{c\eta}] = \frac{1}{\kappa} U_{c\eta\eta} - \frac{f(\alpha)}{1 - \alpha}(U_c - U_d). \quad [51]$$

Making use of the solution of the corresponding problem in the case of a single-phase flow, we put

$$U_c = U e^{\lambda\eta} \cos(\omega t + \mu\eta), \quad [52]$$

$$U_d = [A \cos(\omega t + \mu\eta) + B \sin(\omega t + \mu\eta)]\, e^{\lambda\eta}, \quad [53]$$

and seek the unknown constants λ, μ, A and B by substituting [52] and [53] into [50] and [51]. Note, however, that by contrast to the single-phase case, $\lambda \neq \mu$. Substitution of the assumptions [52] and [53] into [50] and [51] gives a system of four algebraic equations for A, B, λ and μ:

$$(1 + \epsilon)\beta[-(V - V_d)\lambda A - (V - V_d)\mu B + \omega B] = \frac{f(\alpha)}{\alpha}(U - A), \quad [54]$$

$$(1 + \epsilon)\beta[(V - V_d)\mu A - \omega A - (V - V_d)\lambda B] = -\frac{f(\alpha)}{\alpha} B, \quad [55]$$

$$-\beta(V - V_c)\lambda U = \frac{1}{\kappa}(\lambda^2 - \mu^2)U - \frac{f(\alpha)}{1 - \alpha}(U - A), \quad [56]$$

$$\beta[(V - V_c)\mu U - \omega U] = -\frac{2\lambda\mu}{\kappa} U + \frac{f(\alpha)}{1 - \alpha} B. \quad [57]$$

The last two equations give

$$A = U + \frac{1-\alpha}{f(\alpha)}\left(\frac{\mu^2-\lambda^2}{\kappa} - \beta\lambda(V-V_c)\right)U, \quad [58]$$

$$B = \frac{1-\alpha}{f(\alpha)}\left(\frac{2\lambda\mu}{\kappa} + \beta\mu(V-V_c) - \beta\omega\right)U. \quad [59]$$

Introducing [58] and [59] into [55] we obtain an algebraic equation of the second order for λ, with the coefficients being functions of μ. The roots of this equation are real and of opposite signs. Choosing the negative root λ (since $\eta \geq 0$) and introducing it into [54], we

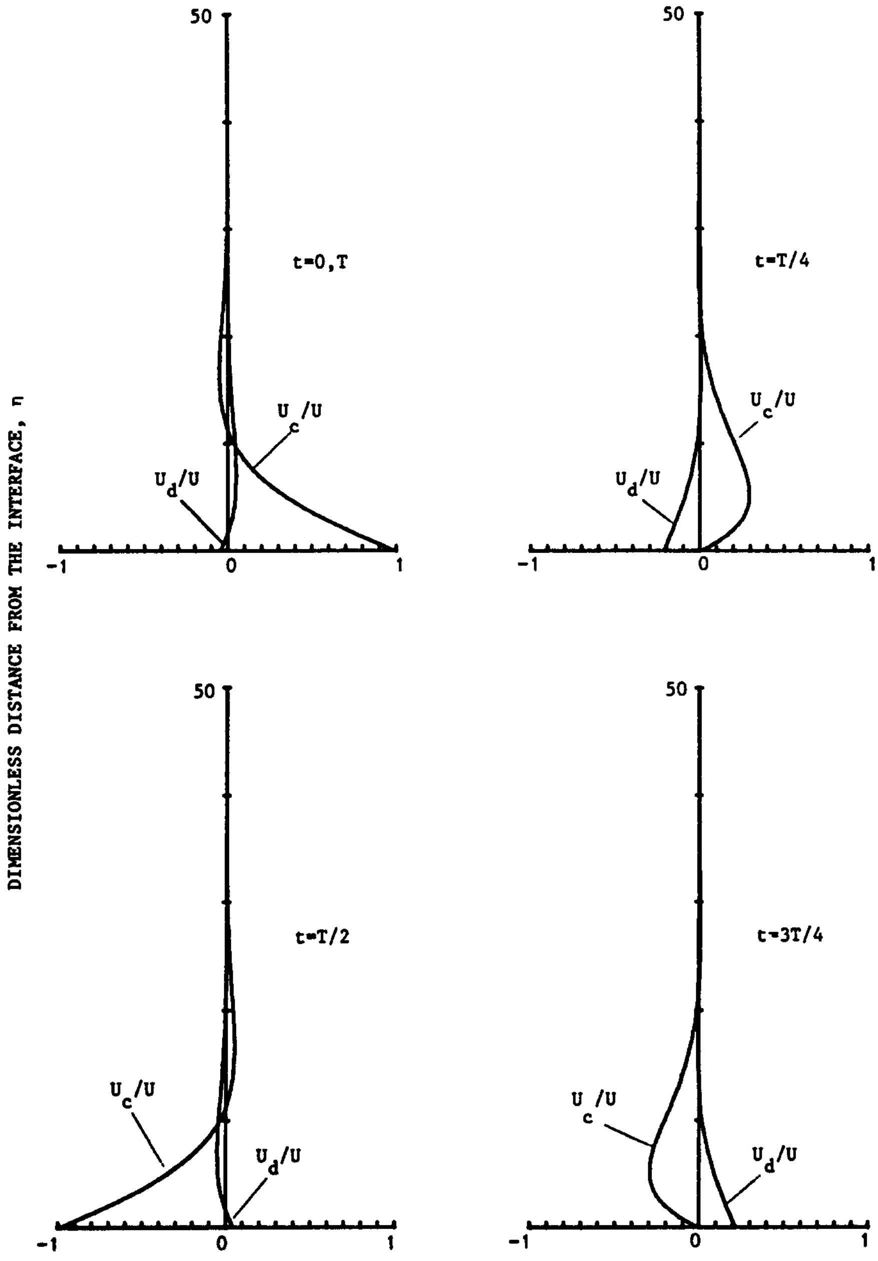

Figure 3. Flow due to an oscillating infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01, \beta = 0.01, \epsilon = 1000, \omega = 1.0$.

eventually obtain an algebraic equation for μ alone:

$$F(\mu) = 0. \qquad [60]$$

Plots of F versus μ for various values of the flow parameters show that [60] has only one real root μ, the value of which is obtained numerically. Plots of the velocity profiles of the phases for various values of the flow parameters are shown in figures 3–9.

5. FLOW IN A MIXTURE INDUCED BY A SUDDENLY ACCELERATED INFINITE PLANE

Finally, we investigate the motion of a semiinfinite region occupied by mixture and bounded by a rigid plane $y = 0$, which is suddenly accelerated to a velocity U parallel to its own direction.

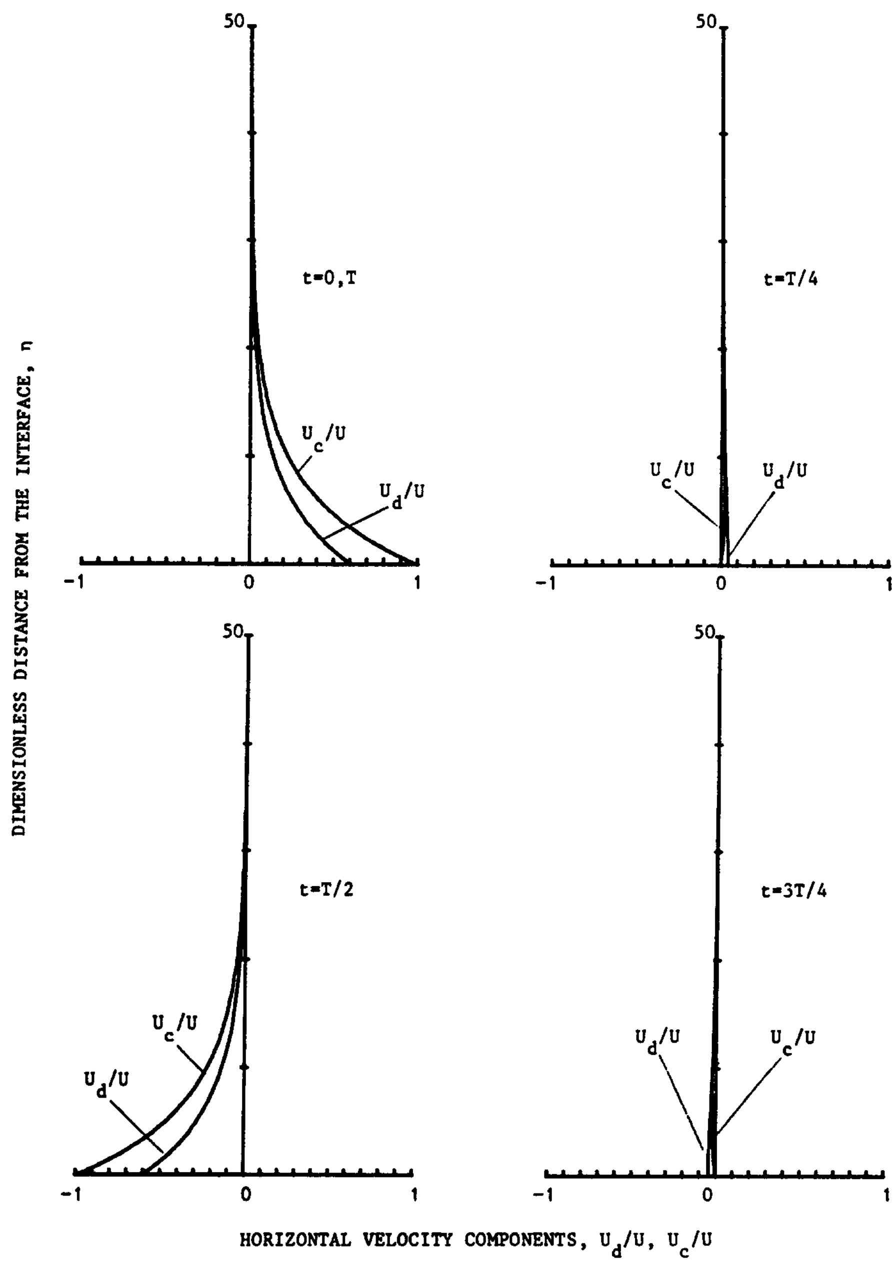

Figure 4. Flow due to an oscillating infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01$, $\beta = 0.01$, $\epsilon = 1000$, $\omega = 0.01$.

We assume as before that the vertical motion of the phases introduced by the gravity field, perpendicular to the rigid boundary, is stationary, and the vertical velocity components of the phases are those given by [30] and [31]. Again we introduce the grouping

$$\eta = y - Vt, \qquad [61]$$

where V is the vertical velocity component of the moving interface between the sediment and the mixture, and is given by [35]. By analogy with the previous section, we assume that the layer of sediment settled on the rigid boundary moves together with the boundary in the horizontal direction and thus obtains a horizontal velocity component equal to U.

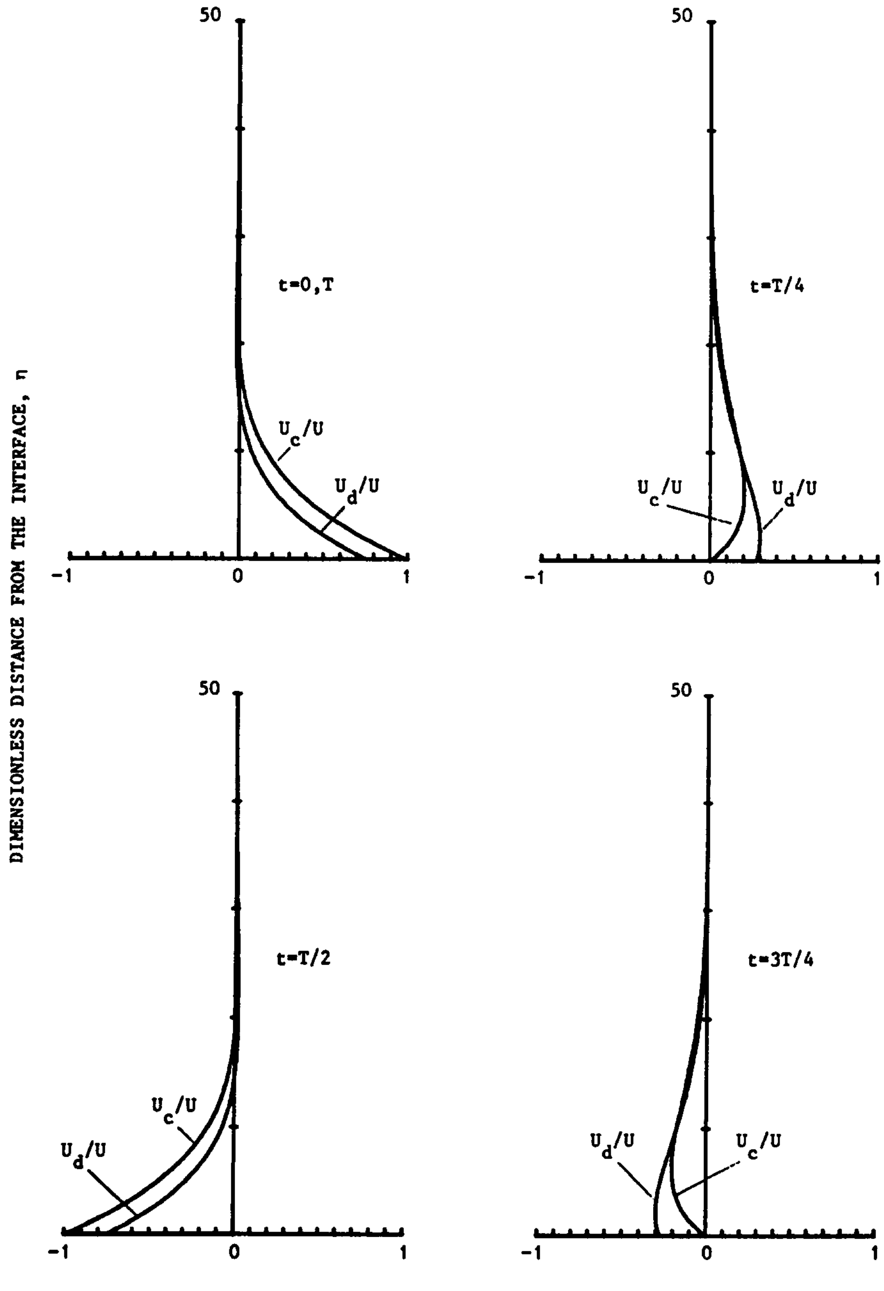

Figure 5. Flow due to an oscillating infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01, \beta = 0.01, \epsilon = 200, \omega = 0.2$.

Selecting as before new independent variables t and η, the x-components of the momentum equations become

$$d: \quad (1+\epsilon)\beta[U_{dt} - (V - V_d)U_{d\eta}] = \frac{f(\alpha)}{\alpha}(U_c - U_d), \qquad [62]$$

$$c: \quad \beta[U_{ct} - (V - V_c)U_{c\eta}] = \frac{1}{\kappa}U_{c\eta\eta} - \frac{f(\alpha)}{1-\alpha}(U_c - U_d), \qquad [63]$$

as in section 4. The boundary conditions in this case are

$$t \leq 0: U_d(t, \eta) = U_c(t, \eta) = 0, \qquad [64]$$

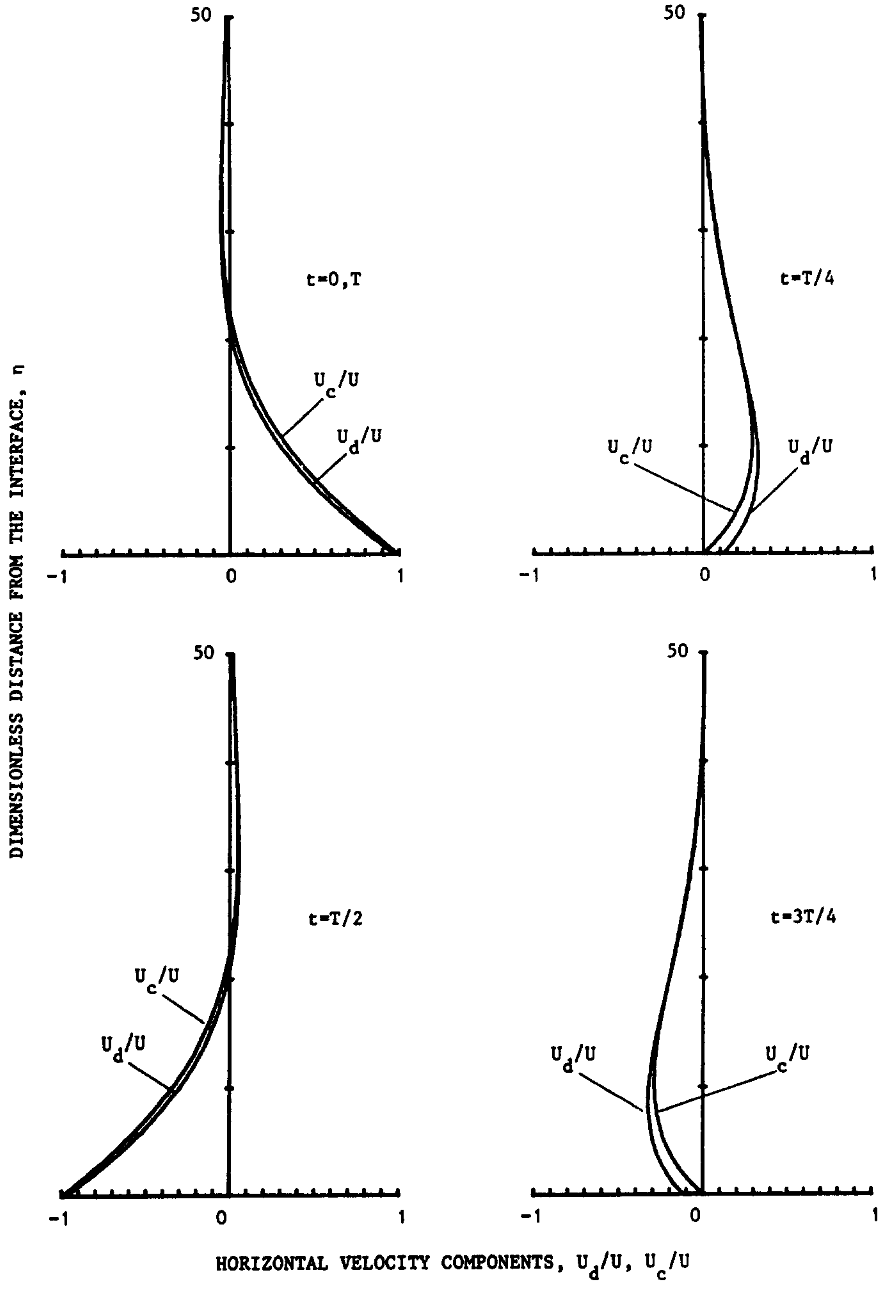

Figure 6. Flow due to an oscillating infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01, \beta = 0.01, \epsilon = 100, \omega = 0.1$.

$$t > 0 : U_c(t, 0) = U, \qquad [65]$$

$$t > 0 : U_c(t, +\infty) = U_d(t, +\infty) = 0. \qquad [66]$$

The Laplace transformed momentum equations are

$$(1 + \epsilon)\beta[s\bar{U}_d - U_d(0, \eta) - (V - V_d)\bar{U}_d'] = \frac{f(\alpha)}{\alpha}(\bar{U}_c - \bar{U}_d), \qquad [67]$$

$$\beta[s\bar{U}_c - U_c(0, \eta) - (V - V_c)\bar{U}_c'] = \frac{1}{\kappa}\bar{U}_c'' - \frac{f(\alpha)}{1 - \alpha}(\bar{U}_c - \bar{U}_d), \qquad [68]$$

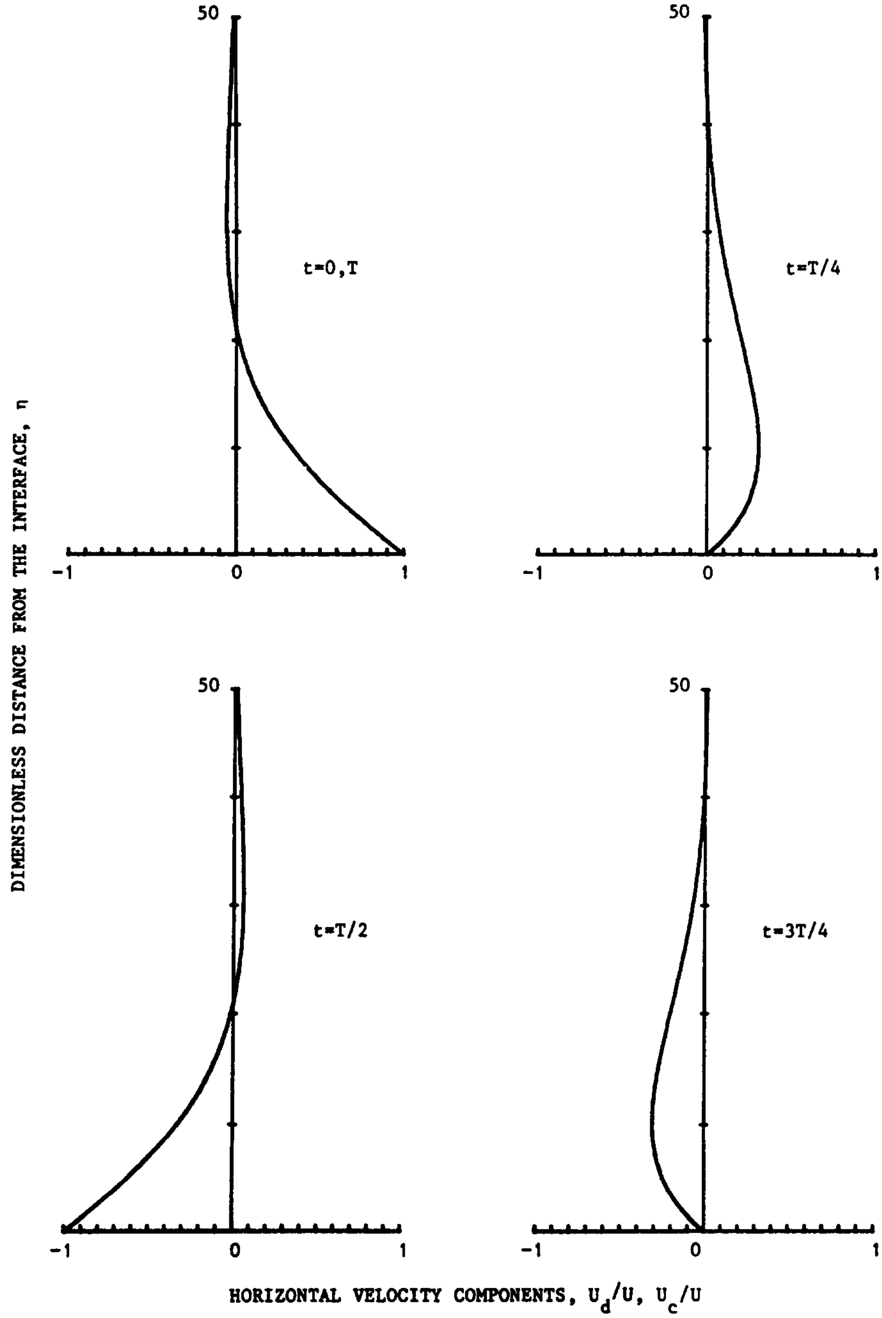

Figure 7. Flow due to an oscillating infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.1, \beta = 0.01, \epsilon = 10, \omega = 0.1$.

where prime denotes differentiation with respect to η, or

$$(1+\epsilon)\beta[s\bar{U}_d - (V - V_d)\bar{U}'_d] = \frac{f(\alpha)}{\alpha}(\bar{U}_c - \bar{U}_d), \quad [69]$$

$$\beta[s\bar{U}_c - (V - V_c)\bar{U}'_c] = \frac{1}{\kappa}\bar{U}''_c - \frac{f(\alpha)}{1-\alpha}(\bar{U}_c - \bar{U}_d), \quad [70]$$

since

$$U_d(0,\eta) = U_c(0,\eta) = 0. \quad [71]$$

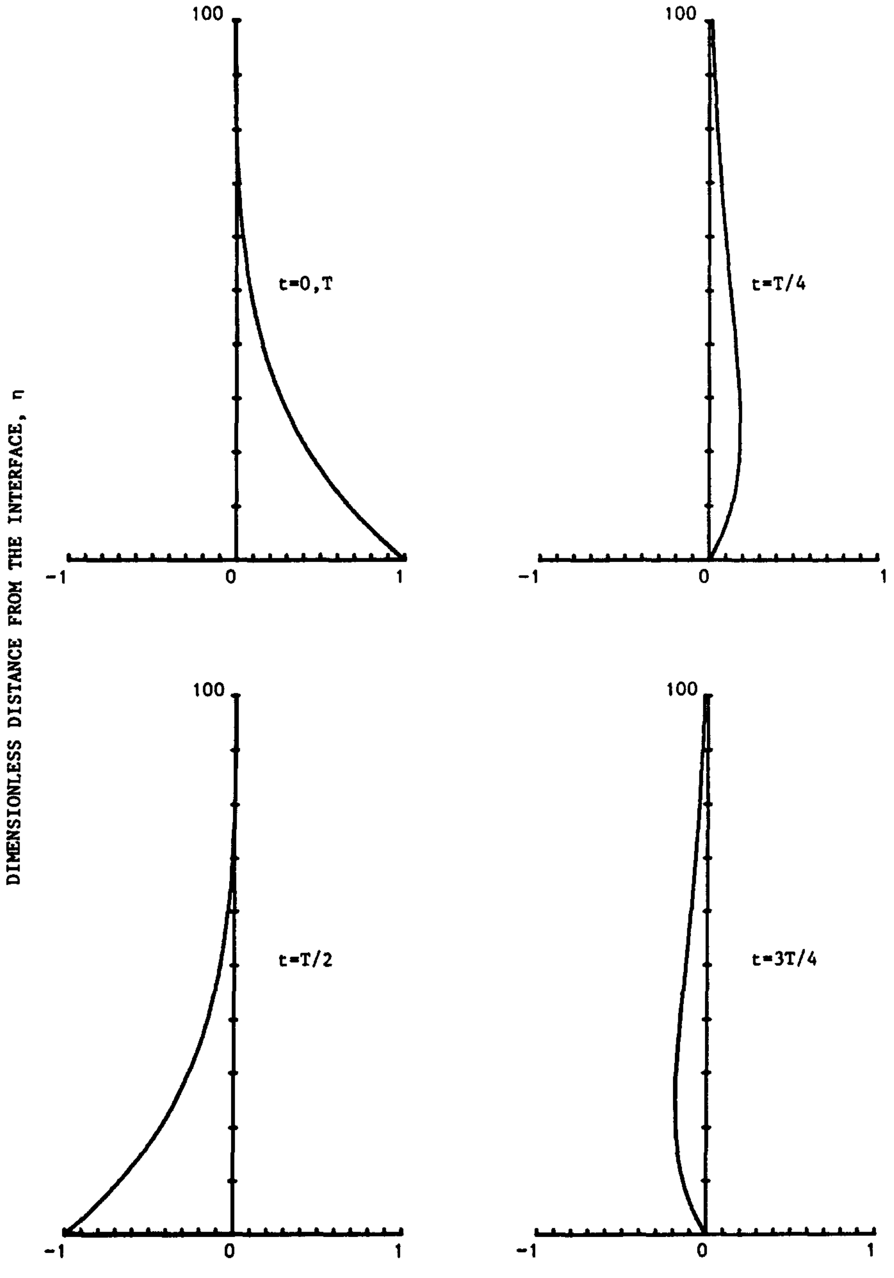

Figure 8. Flow due to an oscillating infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.1$, $\beta = 0.01$, $\epsilon = 10$, $\omega = 0.01$.

The boundary conditions in the transform plane are

$$\overline{U}_c(s, 0) = \frac{U}{s}, \qquad [72]$$

$$\overline{U}_d(s, +\infty) = \overline{U}_c(s, +\infty) = 0. \qquad [73]$$

System [69] and [70] may be reduced to a single ordinary differential equation of the third order with constant coefficients for U_c alone. The general solution of this equation is obtained by considering the characteristic equation, which has three different real roots obtainable in trigonometric form. Two of the roots are positive and the third is negative.

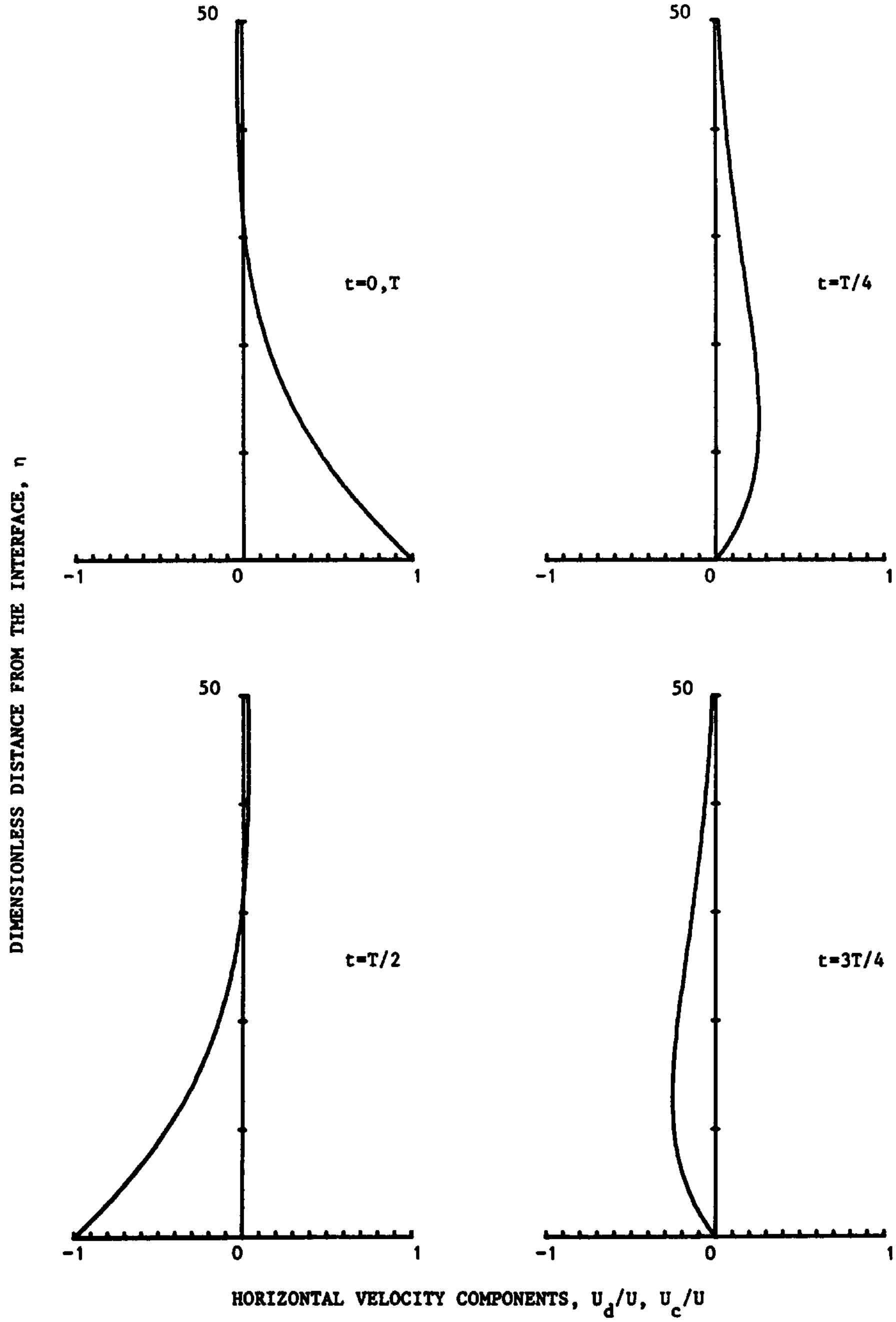

Figure 9. Flow due to an oscillating infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.5, \beta = 0.01, \epsilon = 10, \omega = 0.01$.

Choosing the negative root $\lambda < 0$ (since $\eta \geq 0$) and using [72], we obtain the Laplace transform of the horizontal velocity component of the continuous phase in the form

$$\bar{U}_c = \frac{U}{s} e^{\lambda\eta}. \qquad [74]$$

The Laplace transform of the horizontal velocity component of the dispersed phase is then obtained by means of [70],

$$\bar{U}_d = \frac{U}{s}\left[1 + \frac{1-\alpha}{f(\alpha)}\beta s\right] e^{\lambda\eta} - \frac{U}{s}\beta\frac{1-\alpha}{f(\alpha)}(V - V_c)\lambda e^{\lambda\eta} - \frac{U}{s}\frac{1-\alpha}{\kappa f(\alpha)}\lambda^2 e^{\lambda\eta}, \qquad [75]$$

with $\lambda = \lambda(s)$ being the negative root of the characteristic equation.

Our next step is to obtain inversions of the Laplace transforms $\bar{U}_d$, $\bar{U}_c$. However, since the dependence on the transform parameter s in formulas [74] and [75] is quite complex, $\lambda(s)$ being the root of the third-order algebraic equation obtained in trigonometric form, we will be using approximate numerical methods of inversion.

Figure 10 represents plots of $s\bar{U}_d$, $s\bar{U}_c$ versus the logarithmic transform parameter s for some values of the flow variables and the space coordinate η. Variation of $s\bar{U}_d$, $s\bar{U}_c$ occurs over several decades of $\log_{10}s$. Such functions are said to vary "slowly" with s and may be approximated by series of the form

$$f^*(t) = \sum_{i=1}^{m} c_i \exp(-\sigma_i t), \qquad [76]$$

(see Cost & Becker 1970). Using the approximation procedure described in detail in Cost & Becker (1970), we perform numerical inversions of $\bar{U}_d(s, \eta)$, $\bar{U}_c(s, \eta)$ for various values of the flow variables and thus obtain velocity profiles of the dispersed and continuous phases at different instants, shown in figures 11–16.

6. SUMMARY AND DISCUSSION

Three different cases of a streaming motion of a mixture of solid particles with a fluid were investigated. The horizontal motion of a mixture in these flows, defined by the conditions at the rigid boundary—a horizontal infinite plane—is superimposed on the separational motion of the phases in the vertical direction caused by the gravity field. The considered flows correspond to the three classical problems in a single-phase case, namely, flow over a stationary infinite plane, flow about an oscillating plane, and finally flow in a fluid at rest caused by a suddenly accelerated plane.

In the present investigation, both constituents of a mixture, fluid and solid particles, are treated as two superimposed continua with an interaction defined by the Stokes' drag force, modified by a factor accounting for a finite volume fraction of particles. The analysis of the motion is then performed in a coordinate system, moving in the vertical direction with a velocity of an interface between the layer of a sediment settling on the bottom, the rigid plane, and the region occupied by the mixture, with a given initial concentration of the dispersed phase above. According to Kynch (1952), the concentration of the dispersed phase in the mixture may be assumed to remain constant throughout the separation process. Note, however, that the assumption of constant concentration does not hold in the case of centrifugal separation of a mixture. It was shown by Greenspan (1983) that the concentration of the dispersed phase in a mixture is time dependent in a centrifugal force field. Also, since we are in the present paper dealing with a suspension of solid particles in a continuous carrier fluid and not a mixture of two immiscible fluids, investigated by Greenspan (1983), we are able to specify the form of the correction factor $f(\alpha)$ accounting for the finite volume fraction of the dispersed phase, namely, an expression defined in [8] and obtained by Tam (1969) in the case of spherical particles.

By contrast to the case of single-phase flow, the boundary layer thickness is defined not by the viscosity of the fluid phase but rather by the parameters characterizing correlations of the phases, such as particle Reynolds number, volume concentration and density ratio. It is of importance to remember at this point, however, that the continuum description of a

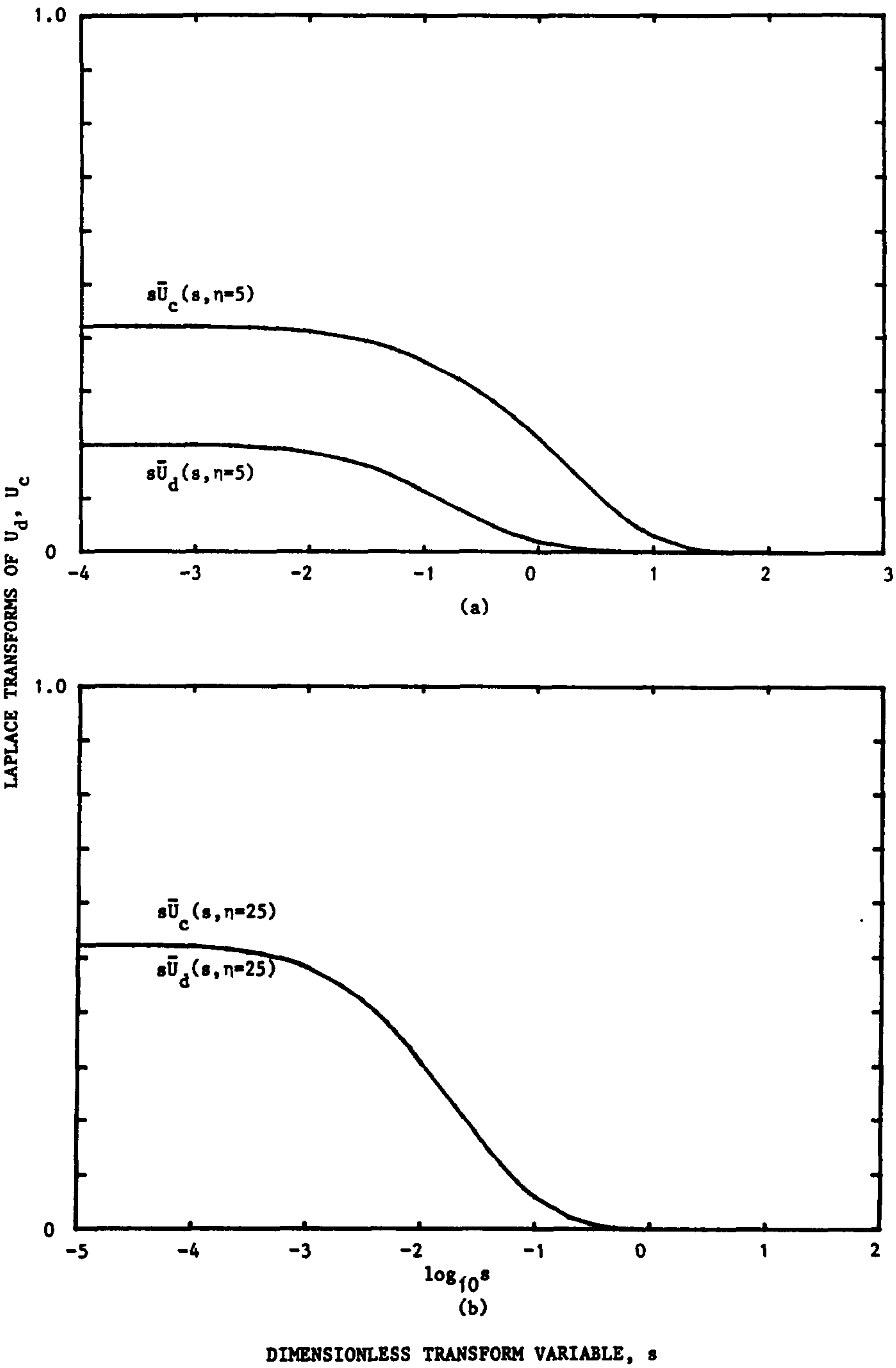

Figure 10. Flow due to a suddenly accelerated infinite plane. Laplace transforms of the horizontal velocities of the phases at different locations versus logarithmic transform parameter. (a) $\alpha = 0.01$, $\beta = 0.01$, $\epsilon = 1000$, $\eta = 5$; (b) $\alpha = 0.1$, $\beta = 0.01$, $\epsilon = 10$, $\eta = 25$.

mixture implies the use of the field variables and balance equations, which are obtained through an averaging procedure over regions containing a sufficiently large number of particles. In the present investigation, the chosen length scale is one particle radius. The obtained solutions of the balance equations thus possess a physical interpretation only if their variations span regions that are large as compared to particle size. We thus select such combinations of the flow variables that give solutions with a scale of variation that is large as compared to the microscale.

In the case of a stationary flow over an infinite rigid plane, the velocity profiles of both phases are obtained analytically, and are shown in figure 2. The velocity profiles of the

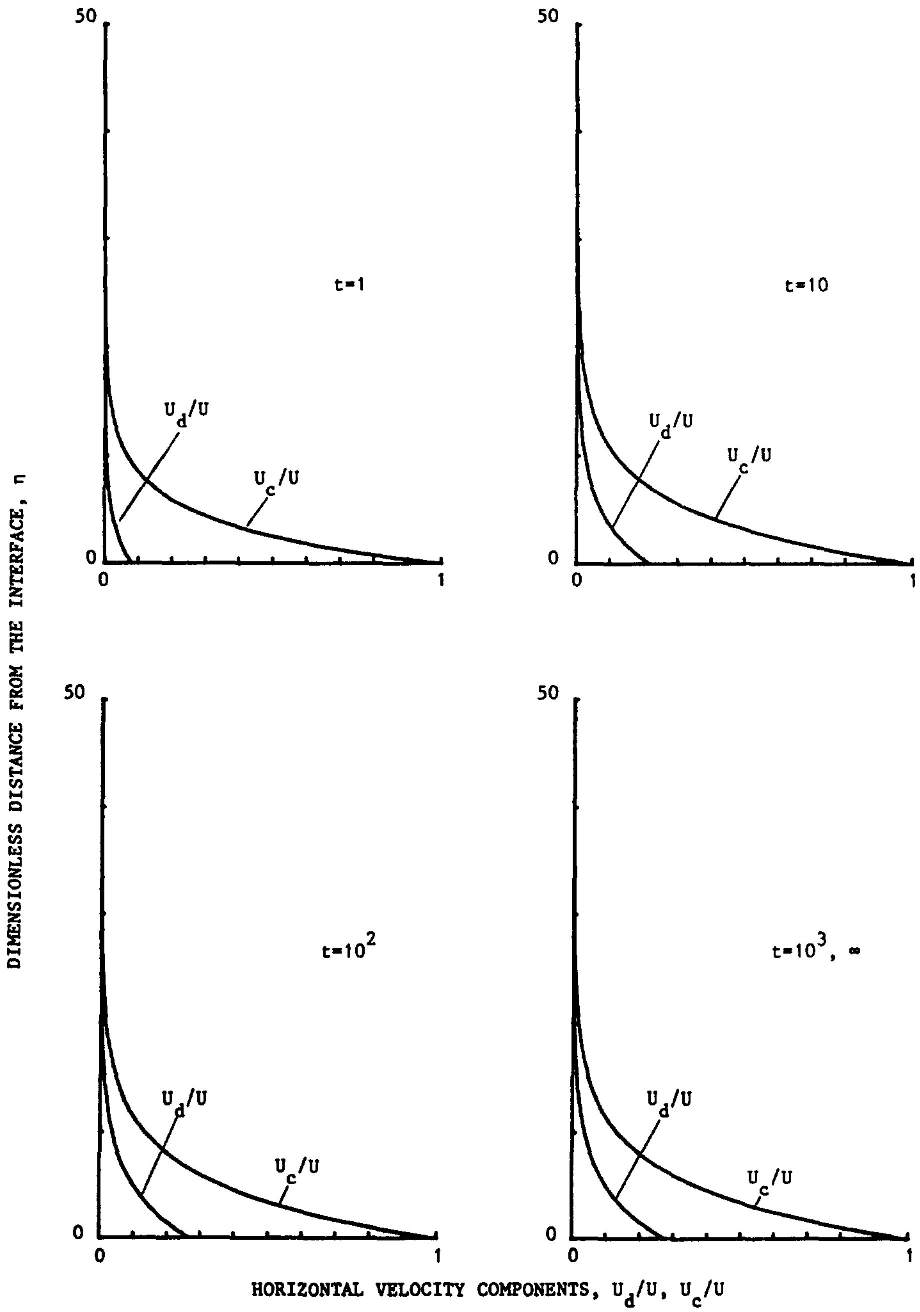

Figure 11. Flow due to a suddenly accelerated infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01$, $\beta = 0.01$, $\epsilon = 2000$.

phases separate in the vicinity of the wall only for mixture flows with high density ratios of the constituents and low values of the concentration of the dispersed phase (see e.g. figure 2a and b). The physical explanation is that because of the great density difference, particle inertia prevents the adjustment of the horizontal velocity component to that of a fluid, which results in a nonzero horizontal velocity component of the dispersed phase at the interface between the mixture and the layer of sediment at the bottom. We assume here, however, that this velocity component is reduced to zero instantly at the moment of contact of a settling particle with the stationary layer of the sediment. At the higher values of the particle concentration and lower density ratios, the velocity profiles of the phases coincide (see figure

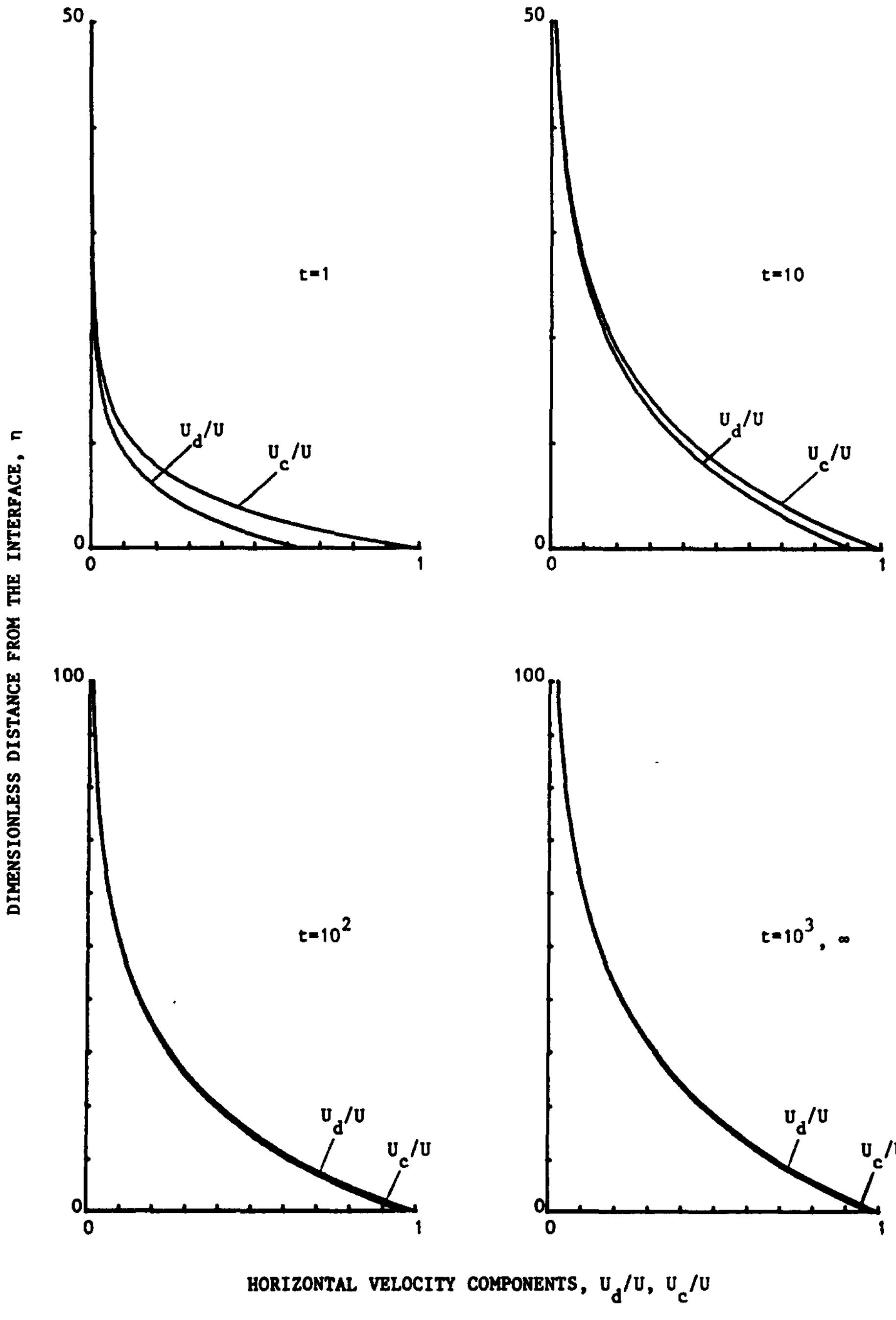

Figure 12. Flow due to a suddenly accelerated infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01, \beta = 0.01, \epsilon = 100$.

2c–f). This means that in this case particles settle with a zero horizontal velocity component on the layer of sediment.

The next problem considered here is the flow of a mixture near an oscillating plane. As before, we assume that the layer of sediment settling on the rigid plane is stationary with respect to its horizontal motion and thus is in this case performing harmonic oscillations in the horizontal direction. The nature of the boundary conditions suggests the consideration of the problem in the coordinate system moving with the vertical velocity component of the interface between the sediment and the mixture. The velocity profiles of both phases in the mixture are then obtained by an assumption similar to that of single-phase flow, and

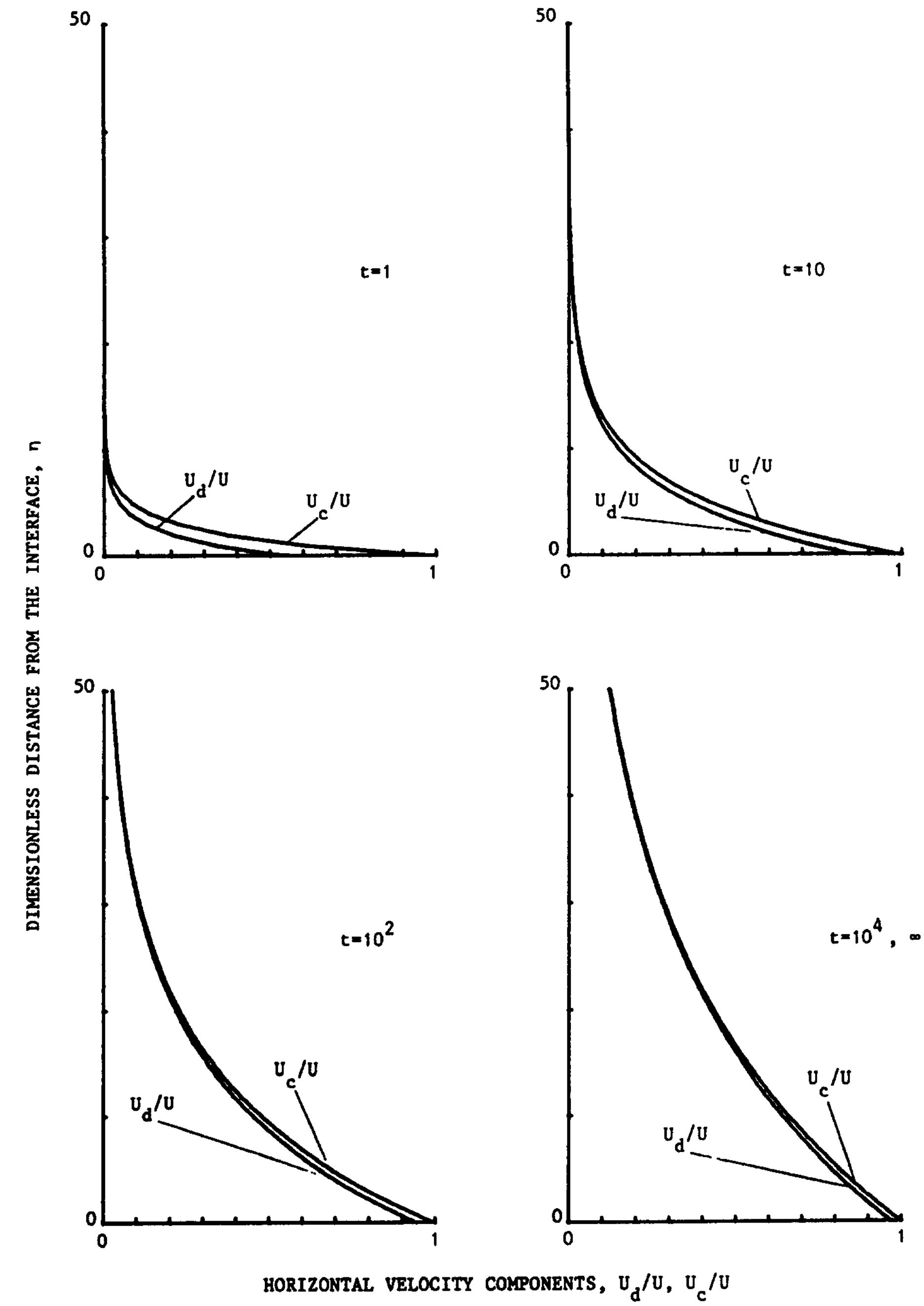

Figure 13. Flow due to a suddenly accelerated infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01, \beta = 0.1, \epsilon = 10$.

represent thus two damped transverse waves propagating into the interior of a mixture. The wave velocity and the wavelength are in this case a function of the particle Reynolds number, concentration, density ratio between the phases and frequency of oscillations. The distance of penetration is in the case of a mixture also defined by the parameters defining correlation of the phases (Reynolds number, concentration, density ratio) and not just the viscosity of the continuous phase and the frequency of oscillations as in the case of single-phase flow. Plots of the velocity profiles of the dispersed and continuous phases for various values of the flow variables are shown in figures 3–9.

Finally, a flow in a stationary mixture induced by a suddenly accelerated rigid plane is

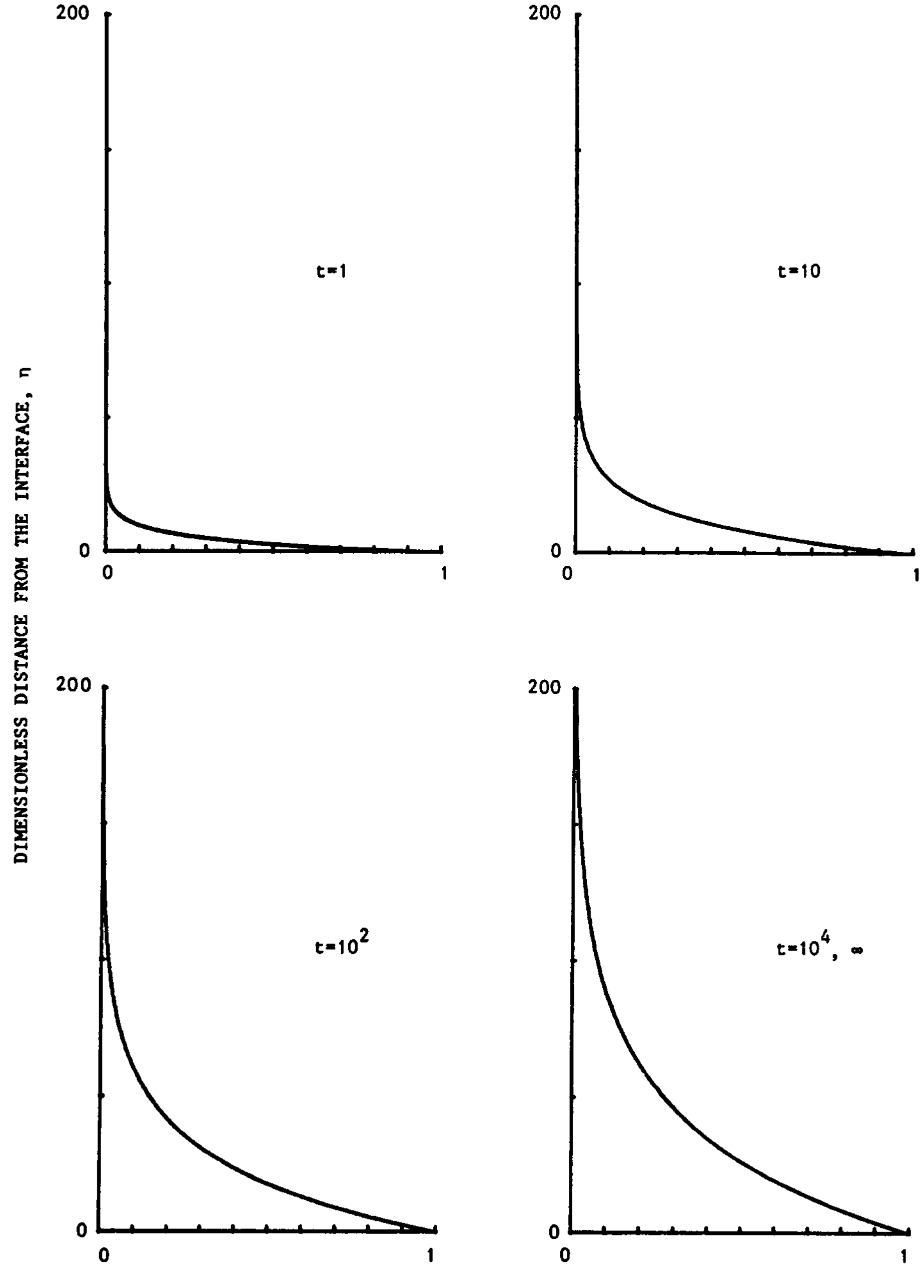

Figure 14. Flow due to a suddenly accelerated infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.1, \beta = 0.01, \epsilon = 10$.

considered. Here again the investigation is performed in a coordinate frame moving upward with a velocity equal to the vertical velocity component of the interface between the sediment and the mixture. Horizontal velocity components of the phases at different instants are obtained by means of the numerical inversion of the solutions to the Laplace transformed momentum equations, and are shown in figures 11–16. In the cases of low concentrations of particles and high density ratios of the constituents, we obtain large differences in the velocities of the phases near the interface. These velocity differences reduce gradually as the velocity profiles of both phases tend to a steady-state distribution, resulting in profiles of a stationary form, propagating into the region occupied by the mixture with the velocity of the interface between the sediment and the mixture. Penetration of the velocity variation at the

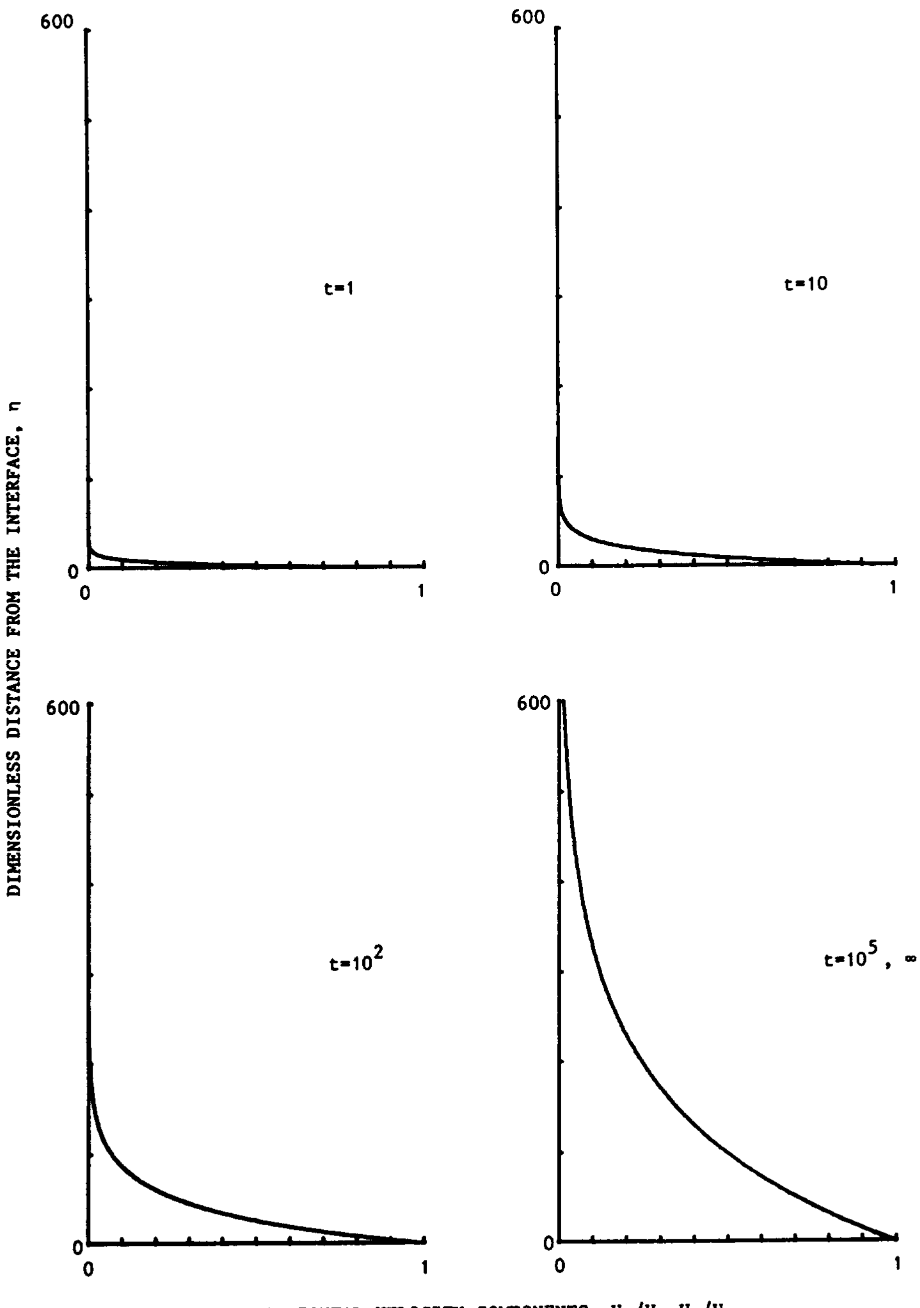

Figure 15. Flow due to a suddenly accelerated infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.4$, $\beta = 0.01$, $\epsilon = 1$.

boundary thus consists of two stages in the case of a mixture flow. In the first stage, the spreading of the velocity variation is a combination of the vertical movement induced by the movement of the interface and development of the velocity profiles toward the steady-state distribution. In the second stage, when the velocity profiles of the phases are fully developed, the velocity variation is transported only by means of the vertical motion of the stationary velocity profiles, propagating upward with the velocity of the interface. Duration of the first stage of the development depends on the particle Reynolds number, density ratio and volume fraction of the dispersed phase and has dimensional values varying from $1.7 \cdot 10^{-1}$ *s* as in figure 11 to $1.7 \cdot 10^{4}$ *s* in figure 16. In the second stage, the depth of penetration of the velocity variation after a time *t* is proportional to the constant velocity of propagation in the vertical direction and is thus linearly proportional to *t*, and not to $t^{1/2}$ as in the case of a single

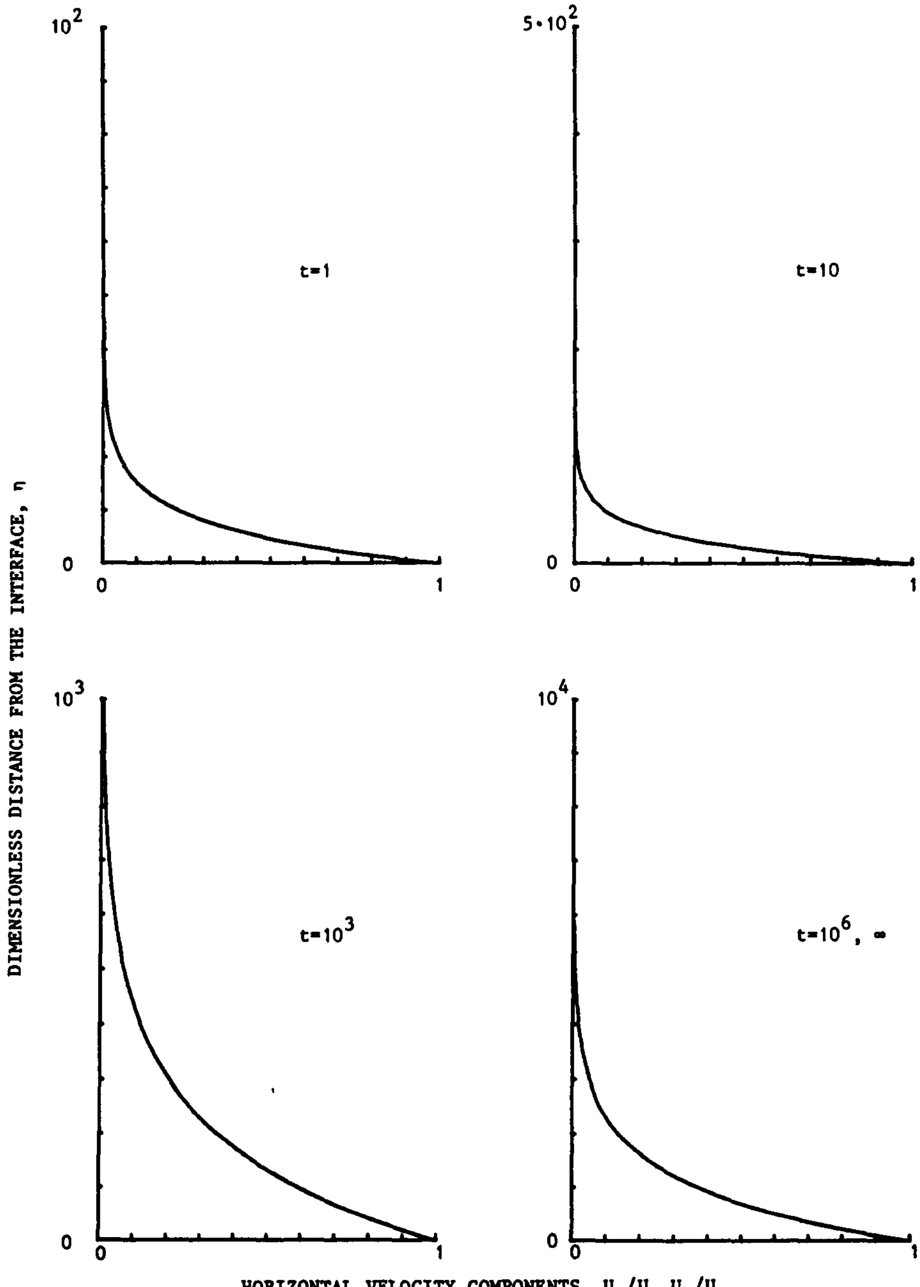

Figure 16. Flow due to a suddenly accelerated infinite plane. Horizontal velocity profiles of the phases at different instants. $\alpha = 0.01$, $\beta = 0.01$, $\epsilon = 1$.

fluid. Figure 11 shows that the depth of penetration of the velocity variation caused by the moving layer of sediment is ~30 particle radii in case of sand (density ~$2 \cdot 10^3$ kg m^{-3}) in air and a dilute particle concentration of 0.01. For sand in water, the depth of penetration of the velocity variation increases drastically to ~600 particle radii when the particle concentration equals 0.4 (figure 15) and to 6000 particle radii at a low concentration of 0.01 (figure 16).

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